

Essential Whittaker functions for $GL(n)$ over a p -adic field

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Abstract

We give a constructive proof of the existence of the essential Whittaker function of a generic representation of $GL(n, F)$, for F a non-archimedean local field, using mirabolic restriction techniques.

Introduction

Let F be nonarchimedean local field, we denote by \mathfrak{O} its ring of integer, and by $\mathfrak{P} = \varpi\mathfrak{O}$ the maximal ideal of this ring, where ϖ is a uniformiser of F . We denote by q the cardinality of $\mathfrak{O}/\mathfrak{P}$. We will denote by $|\cdot|$ the normalised absolute value on F .

We denote $GL(n, F)$ by G_n for $n \geq 1$, by $G_n(\mathfrak{O})$ the group $GL(n, \mathfrak{O})$, and we set $G_0 = \{1\}$. We denote by A_n the torus of diagonal matrices in G_n , and by N_n the unipotent radical of the Borel subgroup of G_n given by upper triangular matrices. For $m \geq 0$, we denote by $H_n(m)$ the subgroup of G_n , given by matrices $\begin{pmatrix} g & V \\ L & t \end{pmatrix}$, for g in $G_{n-1}(\mathfrak{O})$, V in \mathfrak{O}^{n-1} , L with every coefficient in \mathfrak{P}^m , and t in $1 + \mathfrak{P}^m$.

If π is a generic representation of G_n , the essential vector of π was introduced in [J-P-S]. One of its main properties is that, if one calls d the conductor of the representation π , the complex vector space $\pi^{H_n(d)}$ of vectors in π , fixed under $H_n(d)$, is generated by the essential vector of π , and that $\pi^{H_n(k)}$ becomes the null space for $k < d$.

However, to prove its existence, one has to study properties of the Rankin-Selberg integrals associated to the pairs (π, π') , where π' varies through the set of unramified generic representations of G_{n-1} .

We set a few more notations before explaining this.

We denote by ν the positive character $|\cdot| \circ \det$ of G_n . We use the product notation for normalised parabolic induction. We will sometimes, as in [J-P-S], denote by π_{X_1, \dots, X_m} the unramified representation $|\cdot|_F^{s_1} \times \dots \times |\cdot|_F^{s_m}$ of G_m (with X_i identified to q^{-s_i}).

We choose a character θ of $(F, +)$ trivial on \mathfrak{O} but not on \mathfrak{P}^{-1} , and use it to define a non degenerate character, still denoted θ , of the standard unipotent subgroup N_n of G_n , by $\theta(n) = \theta(\sum_{i=1}^{n-1} n_{i, i+1})$.

We denote by \hat{G} the group of locally constant complex characters of a locally compact totally disconnected group G .

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Let π and π' be generic representations of G_n and G_m respectively, with $n \geq m$, and denote by $W(\pi, \theta)$ and $W(\pi', \theta^{-1})$ their respective Whittaker models with respect to θ and θ^{-1} .

When $n > m$, and W and W' are respectively in $W(\pi, \theta)$ and $W(\pi', \theta^{-1})$, we denote by $I(W, W', s)$

the integral $\int_{N_m \backslash G_m} W \begin{pmatrix} g & \\ & I_{n-m} \end{pmatrix} W'(g) |det(g)|_F^{s-(n-m)/2} dg$.

When $n = m$, and W and W' are respectively in $W(\pi, \theta)$ and $W(\pi', \theta^{-1})$, ϕ is in $\mathcal{C}_c^\infty(F^n)$, and η is the row vector $(0, \dots, 0, 1)$ in the space $\mathcal{M}(1, n, F)$ of row matrices 1 by n with entries in F , we denote by $I(W, W', \phi, s)$ the integral $\int_{N_n \backslash G_n} W(g) W'(g) \phi(\eta g) |det(g)|_F^s dg$.

These integrals converge absolutely for $Re(s)$ large, and define elements of $\mathbb{C}(q_F^{-s})$.

To define the essential vector of π , one needs to show (as in [J-P-S]) the following theorem:

Theorem. *Let π be a generic representation of G_n with Whittaker model $W(\pi, \theta)$, then there exists in $W(\pi, \theta)$ a unique $G_{n-1}(\mathfrak{O})$ -invariant function W_π^{ess} , such that for every unramified generic representation π' of G_{n-1} , with normalised spherical function $W_{\pi'}^0$ in $W(\pi', \theta^{-1})$, one has the equality $I(W_\pi^{ess}, W_{\pi'}^0, s) = L(\pi, \pi', s)$.*

Using this theorem, it is then shown in [J-P-S], playing with the functional equation, that the $W(\pi, \theta)^{H_n(d)}$ is a complex line spanned by W_π^{ess} , and that $W(\pi, \theta)^{H_n(k)}$ is zero for $k < d$.

Remark. The reason why we got interested in reproving the existence of such a vector is the following. In [J-P-S], the uniqueness of such a vector is proved. The proof of the existence is valid only for generic representations π appearing as subquotients of representations parabolically induced by ramified characters of $GL(1, F)$ and cuspidal representations of $GL(r, F)$ for $r \geq 2$, i.e. generic representations with L -function equal to one.

Before we explain this, let us mention that Jacquet (see [J]) found a simple fix for the proof of [J-P-S], so that the motivation of writing our note is really to give a constructive proof of the existence of this vector, which provides a nice application of the techniques developed in [C-P].

In [J-P-S], the following is showed: for fixed W in $W(\pi, \theta)$, the function

$$P(W, X_1, \dots, X_{n-1}) = I(W, W_{\pi_{X_1, \dots, X_{n-1}}}^0, 0) / L(\pi, \pi_{X_1, \dots, X_{n-1}}, 0)$$

belongs to the ring $\mathbb{C}[X_1, \dots, X_{n-1}, X_1^{-1}, \dots, X_{n-1}^{-1}]^{S_{n-1}}$ of symmetric Laurent polynomials in $n-1$ variable. It is also shown that the existence of the essential vector is equivalent to the fact that the vector space $I(\pi) = \{P(W, X_1, \dots, X_{n-1}), W \in W(\pi, \theta)\}$, which is actually an ideal, is equal to $\mathbb{C}[X_1, \dots, X_{n-1}, X_1^{-1}, \dots, X_{n-1}^{-1}]^{S_{n-1}}$.

The argument used to prove it goes like this:

For W well chosen, $P(W, X_1, \dots, X_{n-1})$ is equal to $1/L(\pi, \pi_{X_1, \dots, X_{n-1}}, 0) = \prod_{i=1}^{n-1} 1/L(\pi, \pi_{X_i}, 0)$, where π_{X_i} is nothing else than the character $|\cdot|^{s_i}$ of F^* . We denote by $Q(X)$ the element $1/L(\pi, \pi_X, 0) = 1/L(\pi, X)$ of $\mathbb{C}[X]$, so that $P(W, X_1, \dots, X_{n-1}) = \prod_{i=1}^{n-1} Q(X_i)$.

But because of the functional equation of the L -function $L(\pi, \pi_{X_1, \dots, X_{n-1}}, s)$, one shows that $I(\pi)$ also contains the product $\prod_{i=1}^{n-1} Q'(q_F^{-1} X_i^{-1})$, where $Q'(X) = 1/L(\pi^\vee, \pi_X, 0) = 1/L(\pi^\vee, X)$. A lemma at the beginning of the paper then shows that $Q'(q^{-1} X^{-1})$ and $Q(X)$ are prime to one another in $\mathbb{C}[X, X^{-1}]$.

They deduce from this that no maximal ideal

$$I_{x_1, \dots, x_{n-1}} = \{R \in \mathbb{C}[X_1, \dots, X_{n-1}, X_1^{-1}, \dots, X_{n-1}^{-1}], R(x_1, \dots, x_{n-1}) = 0\}$$

for (x_1, \dots, x_{n-1}) in \mathbb{C}^{*n-1} contains $\prod_{i=1}^{n-1} Q'(q^{-1} X_i^{-1})$ and $\prod_{i=1}^{n-1} Q(X_i)$ together, which implies the result.

But this last step is false as soon as $n \geq 3$, and there are a and b in \mathbb{C}^* such that $Q(a) = Q'(q^{-1} b^{-1}) = 0$, because then both products belong to any ideal $I_{a, b, \dots, x_{n-1}}$. This is the case as

soon as the degree $d^\circ(Q)$ of Q satisfies $d^\circ(Q) \geq 1$.

However, using the functional equation of the L -function $L(\pi, \pi_{X_1, \dots, X_{n-1}})$, and the cyclicity of the spherical Whittaker function in $\pi_{X_1, \dots, X_{n-1}}$, Jacquet noticed that one can find for every (x_1, \dots, x_{n-1}) in \mathbb{C}^{*n-1} , a polynomial in $I(\pi)$, taking the value 1 at (x_1, \dots, x_{n-1}) , so $I(\pi)$ is indeed equal to $\mathbb{C}[X_1, \dots, X_{n-1}, X_1^{-1}, \dots, X_{n-1}^{-1}]^{S_{n-1}}$.

The idea of our proof is the following.

We recall (Theorem 9.7 of [Z]), that every generic representation π of $GL(n, F)$ can be written uniquely, up to permutation of the terms in the product, as a product of non-linked segments

$$[\nu^{-k_1(\pi)-1}\rho_1(\pi), \dots, \rho_1(\pi)] \times \dots \times [\nu^{-k_t(\pi)-1}\rho_t(\pi), \dots, \rho_t(\pi)].$$

Here $\rho_i(\pi)$ denotes a cuspidal representation of $G_{r_i(\pi)}$, and $[\nu^{-k_i(\pi)-1}\rho_i(\pi), \dots, \rho_i(\pi)]$ is the unique irreducible quotient of $\nu^{-k_i(\pi)-1}\rho_i(\pi) \times \dots \times \rho_i(\pi)$, so that $\sum_i k_i(\pi)r_i(\pi) = n$.

If $\pi' = \mu_1 \times \dots \times \mu_{n-1}$ is an unramified generic representation of G_{n-1} , then, from [J-P-S 2] or [C-P] (notice that the results in these two references do not rely on results of [J-S]), one has the equality of Rankin-Selberg L -functions $L(\pi, \pi', s) = \prod_{i,j} L(\rho_i(\pi), \mu_j, s)$.

We notice that $L(\rho_i(\pi), \mu_j, s)$ is equal to 1 unless $\rho_i(\pi)$ is an unramified character of G_1 . Hence, by what we said just before, one has $L(\pi, \pi', s) = \prod_{\{i, \rho_i(\pi) \in \widehat{F^*}/\mathfrak{D}^*, j\}} L(\rho_i(\pi), \mu_j, s)$.

Denote by χ_1, \dots, χ_r the elements of the set $\{j, \rho_j(\pi) \in \widehat{F^*}/\mathfrak{D}^*\}$, counted with multiplicity, so that the L function $L(\pi, \pi', s)$ is equal to $\prod_{i,j} L(\chi_i, \mu_j, s)$.

If $r = n$, the existence of the essential vector is known, it is just the spherical vector W_π^0 . Hence we're left with the case $r \leq n-1$.

First, we will assume that the G_r -module $\pi_u = \chi_1 \times \dots \times \chi_r$ is an irreducible submodule of the Bernstein-Zelevinsky derivative $\pi^{(n-r)}$ (it is always a subquotient). Then π_u is generic and the L function of the pair (π, π') is equal to $L(\pi', \pi_u, s)$. It is then proved in Proposition 2.3 of [J-S] and section 1 (equality (3)) of [J-S 2], that $L(\pi', \pi_u, s)$ is equal to

$$I(W_{\pi'}^0, W_{\pi_u}^0, \mathbf{1}_{\mathfrak{D}^{n-1}}, s) = \int_{A_{n-1}} W_{\pi'}^0(a) W_{\pi_u}^0(a) \mathbf{1}_{\mathfrak{D}}(a_{n-1}) \delta_{N_{n-1}}(a)^{-1} |det(a)|^s d^*a,$$

when $r = n-1$, or to

$$I(W_{\pi'}^0, W_{\pi_u}^0, s) = \int_{A_r} W_{\pi'}^0 \left(\begin{smallmatrix} a & \\ & I_{n-1-r} \end{smallmatrix} \right) W_{\pi_u}^0(a) \delta_{N_r}(a)^{-1} |det(a)|^{s-(n-1-r)/2} d^*a$$

otherwise (see the beginning of Section 1 for the definition of the modulus character δ_{N_r}). In the following sections, we will show how to produce an element W in $W(\pi, \theta)^{G_{n-1}(\mathfrak{D})}$, such that $W(diag(a_1, \dots, a_r, a_{r+1}, \dots, a_{n-1}, 1))$ is equal to

$$\frac{\delta_{N_{n-1}} \left(\begin{smallmatrix} a & \\ & I_{n-1-r} \end{smallmatrix} \right)}{\delta_{N_r}(a)} |det(a)|^{(r-n+2)/2} W_{\pi_u}^0(diag(a_1, \dots, a_r)) \mathbf{1}_{\mathfrak{D}}(a_r) \prod_{r < i < n} \mathbf{1}_{\mathfrak{D}^*}(a_i).$$

Then for such a W , remembering that $W_{\pi'}^0 \left(\begin{smallmatrix} a & \\ & I_{n-1-r} \end{smallmatrix} \right)$ vanishes, as soon as $|a_r| > 1$ (see [S]) for $r < n-1$, one has either the equality

$$\begin{aligned} I(W, W_{\pi'}^0, s) &= \int_{A_{n-1}} W \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix} \right) W_{\pi'}^0(a) \delta_{N_{n-1}}(a)^{-1} |det(a)|^{s-1/2} d^*a \\ &= \int_{A_{n-1}} W_{\pi_u}^0(a) W_{\pi'}^0(a) \mathbf{1}_{\mathfrak{D}}(a_{n-1}) \delta_{N_{n-1}}(a)^{-1} |det(a)|^s d^*a \end{aligned}$$

if $r = n - 1$, and similarly

$$I(W, W_{\pi'}^0, s) = \int_{A_r} W_{\pi_u}^0(a) W_{\pi'}^0 \left(\begin{smallmatrix} a & \\ & I_{n-1-r} \end{smallmatrix} \right) \delta_{N_r}(a)^{-1} |\det(a)|^{s-(n-r)/2} d^*a$$

otherwise, which is what we want.

The construction of such a W will rely on the study of the derivatives of π , for example, one of the main properties of W will be that its image in $\pi^{(n-r)}$ is equal to $W_{\pi_u}^0$.

For the general case, we write π as a product

$$[\nu^{-k_1(\pi)-1} \rho_1(\pi), \dots, \rho_1(\pi)] \times \dots \times [\nu^{-k_t(\pi)-1} \rho_t(\pi), \dots, \rho_t(\pi)]$$

and introduce the representation

$$\pi^v = \nu^{v_1} [\nu^{-k_1(\pi)-1} \rho_1(\pi), \dots, \rho_1(\pi)] \times \dots \times \nu^{v_t} [\nu^{-k_t(\pi)-1} \rho_t(\pi), \dots, \rho_t(\pi)]$$

for $v = (v_1, \dots, v_t)$ in \mathcal{D}^t , where \mathcal{D} is the complex algebraic variety $\mathbb{C}/\frac{2i\pi}{Ln(q)}\mathbb{Z}$. We will see that for v in a Zariski open subset of \mathcal{D}^t , π_u^v is an irreducible submodule of $\pi^{v(n-r)}$, so that the vector $W_{\pi^v}^{ess}$ is well defined, and we will then extend the map $v \mapsto W_{\pi^v}^{ess}$ to v in \mathcal{D}^r . The essential vector of π will then be $W_{\pi^0}^{ess}$.

We will start with a section about derivatives, and how they preserve sphericity.

1 Mirabolic restriction and sphericity

For $n \geq 2$ we denote by U_n the group of matrices of the form $\begin{bmatrix} I_{n-1} & V \\ & 1 \end{bmatrix}$.

For $n > k \geq 1$, the group G_k embeds naturally in G_n , and is given by matrices of the form $\text{diag}(g, I_{n-k})$; we denote by Z_k its center; we parametrise it by F^* using the morphism $\beta_k : z_k \mapsto \text{diag}(z_k I_k, I_{n-k})$. Hence the maximal torus A_n of G_n is the direct product $Z_1 \cdot Z_2 \dots Z_{n-1} \cdot Z_n$. We will sometimes (but not always) omit the β_k 's in this parametrisation and write $(z_1 \dots z_n)$ or (z_1, \dots, z_n) for the element $\beta_1(z_1) \dots \beta_n(z_n)$ of A_n . Notice that the i -th simple root α_i has the property that $\alpha_i(z_1, \dots, z_n) = z_i$.

We denote by P_n the mirabolic subgroup $G_{n-1}U_n$ of G_n . If one sees P_{n-1} as a subgroup of G_{n-1} itself embedded in G_n as before, then P_{n-1} is the normaliser of $\theta|_{U_n}$ in G_{n-1} (i.e. if $g \in G_{n-1}$, then $\theta(g^{-1}ug)$ for all $u \in U_n$ if and only if $g \in P_{n-1}$).

We set some notations before recalling a few facts about derivatives.

Following Bernstein and Zelevinsky's conventions, when G is an l -group (locally compact totally disconnected group), we denote by $\text{Alg}(G)$ the category of smooth complex G -modules. If (π, V) belongs to $\text{Alg}(G)$, H is a closed subgroup of G , and χ is a character of H , we denote by $V(H, \chi)$ the subspace of V generated by vectors of the form $\pi(h)v - \chi(h)v$ for h in H and v in V . This space is actually stable under the action of the subgroup $N_G(\chi)$ of the normalizer $N_G(H)$ of H in G , which fixes χ .

We denote by δ_H the positive character of $N_G(H)$ such that if μ is a right Haar measure on H , and int is the action given by $(\text{int}(n)f)(h) = f(n^{-1}hn)$, of $N_G(H)$ smooth functions f with compact support on H , then $\mu \circ \text{int}(n) = \delta_H(n)\mu$ for n in $N_G(H)$.

Let's give an example that we will use later. The group N_n is the product $U_2 \dots U_n$ for $n \geq 2$, and one checks that if du_i is a Haar measure on U_i , then $dn = du_n \dots du_2$ is a Haar measure on N_n . Parametrizing A_n by the Z_i 's, one deduces the relation

$$\delta_{N_n}(z_1, \dots, z_{n-1}, z_n) = \delta_{N_n}(z_1, \dots, z_{n-1}, 1) = \prod_{i=1}^{n-1} \delta_{U_{i+1}}(z_1, \dots, z_i). \quad (1)$$

The space $V(H, \chi)$ is $N_G(\chi)$ -stable. Thus, if L is a closed-subgroup of $N_G(\chi)$, and μ is a (smooth) character of L , the quotient $V_{H, \chi} = V/V(H, \chi)$ (that we simply denote by V_H when χ is trivial) becomes a smooth L -module for the action $l.(v + V(H, \chi)) = \mu(l)\pi(l)v + V(H, \chi)$ of L on $V_{H, \chi}$. We also denote by V^H the subspace of vectors of V fixed by H ; if H is compact, the functor $V \mapsto V^H$ from $\text{Alg}(G)$ to $\text{Alg}(\{1\})$ is exact.

If H is a closed subgroup of an l -group G , and (ρ, W) belongs to $\text{Alg}(H)$, we define the objects $(\text{ind}_H^G(\rho), V_c = \text{ind}_H^G(W))$ and $(\text{Ind}_H^G(\rho), V = \text{Ind}_H^G(W))$ of $\text{Alg}(G)$ as follows. The space V is the space of smooth functions from G to W , fixed under right translation by the elements of a compact open subgroup U_f of G , and satisfying $f(hg) = \rho(h)f(g)$ for all h in H and g in G . The space V_c is the subspace of V , consisting of functions with support compact mod H , in both cases, the action of G is by right translation on the functions.

We define the following functors:

- The functor Φ^- from $\text{Alg}(P_k)$ to $\text{Alg}(P_{k-1})$ such that, if (π, V) is a smooth P_k -module, $\Phi^-V = V_{U_k, \theta}$, and P_{k-1} acts on $\Phi^-(V)$ by $\Phi^-\pi(p)(v + V(U_k, \theta)) = \delta_{U_k}(p)^{-1/2}\pi(p)(v + V(U_k, \theta))$.
- The functor Φ^+ from $\text{Alg}(P_{k-1})$ to $\text{Alg}(P_k)$ such that, for π in $\text{Alg}(P_{k-1})$, one has $\Phi^+\pi = \text{ind}_{P_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \theta)$.
- The functor $\hat{\Phi}^+$ from $\text{Alg}(P_{k-1})$ to $\text{Alg}(P_k)$ such that, for π in $\text{Alg}(P_{k-1})$, one has $\hat{\Phi}^+\pi = \text{Ind}_{P_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \theta)$.
- The functor Ψ^- from $\text{Alg}(P_k)$ to $\text{Alg}(G_{k-1})$, such that if (π, V) is a smooth P_k -module, $\Psi^-V = V_{U_k, 1}$, and G_{k-1} acts on $\Psi^-(V)$ by $\Psi^-\pi(g)(v) + V(U_k, 1) = \delta_{U_k}(g)^{-1/2}\pi(p)(v + V(U_k, 1))$.
- The functor Ψ^+ from $\text{Alg}(G_{k-1})$ to $\text{Alg}(P_k)$, such that for π in $\text{Alg}(G_{k-1})$, one has $\Psi^+\pi = \text{ind}_{G_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes 1) = \delta_{U_k}^{1/2}\pi \otimes 1$.

These functors have the following properties which can be found in [B-Z]:

Proposition 1.1. *a) The functors Φ^- , Φ^+ , Ψ^- , and Ψ^+ are exact.*

b) Ψ^- is left adjoint to Ψ^+ .

b') Φ^- is left adjoint to $\hat{\Phi}^+$.

c) $\Phi^-\Psi^+ = 0$ and $\Psi^-\Phi^+ = 0$.

d) $\Psi^-\Psi^+ \simeq \text{Id}$ and $\Phi^-\Phi^+ \simeq \text{Id}$.

e) One has the exact sequence $0 \rightarrow \Phi^+\Phi^- \rightarrow \text{Id} \rightarrow \Psi^+\Psi^- \rightarrow 0$.

Following [C-P], if τ belongs to $\text{Alg}(P_n)$, we will denote $(\Phi^-)^{k-1}\tau$ by $\tau_{(k)}$, and as usual, $\tau^{(k)}$ will be defined as $\Psi^-\tau_{(k)}$.

Because of e), τ has a natural filtration of P_n -modules $0 \subset \tau_{n-1} \subset \dots \subset \tau_0 = \tau$, where $\tau_k = \Phi^{+k}\Phi^{-k}\tau$. We will use the notation $\tau_{(k), i}$ for $(\tau_{(k)})_i$. The following observation is just a restatement of the definitions:

Lemma 1.1. *If τ belongs to $\text{Alg}(P_n)$, then $\tau_k = \Phi^+(\tau_{(1), k-1})$ for $k \geq 1$.*

Property d) says that Φ^- sends $\Phi^+\tau$ surjectively onto a P_n -module isomorphic to τ . Writing $\Phi^+\tau$ as $\text{ind}_{P_n U_{n+1}}^{P_{n+1}}(\delta_{U_{n+1}}^{1/2}\tau \otimes \theta)$, we want to make the map Φ^- explicit between $\text{ind}_{P_n U_{n+1}}^{P_{n+1}}(\delta_{U_{n+1}}^{1/2}\tau \otimes \theta)$ and τ .

Proposition 1.2. *If τ belongs to $\text{Alg}(P_n)$, then Φ^- identifies with the map $f \mapsto f(I_{n+1})$ from $\Phi^+\tau$ to τ .*

Proof. Up to a good choice of the isomorphism of P_n -modules between $\Phi^-\Phi^+\tau$ and τ , we only need to check that $\Phi^+\tau(U_{n+1}, \theta)$ is equal to the space $\{f \in \Phi^+\tau, f(I_{n+1}) = 0\}$, which is an easy adaptation of the Proposition 2.1. of [C-P]. \square

We recall that for $k \geq 2$, as a consequence of the Iwasawa decomposition, any element g of G_k can be written in the form pzc with p in P_k , z in Z_k , and c in $K = G_k(\mathfrak{O})$. We now notice that the restriction of Φ^- to $(\Phi^+\tau)^{P_{n+1}(\mathfrak{O})}$ is surjective onto $\tau^{P_n(\mathfrak{O})}$.

Proposition 1.3. *The map $f \mapsto f(I_{n+1})$ from $(\Phi^+\tau)^{P_{n+1}(\mathfrak{O})}$ to $\tau^{P_n(\mathfrak{O})}$ is surjective.*

Proof. Let v_0 be a vector in the space of τ which is $P_n(\mathfrak{O})$ -invariant, then one checks that the function f defined by $f\left(\begin{smallmatrix} zp^k & x \\ & 1 \end{smallmatrix}\right) = \delta_{U_k}^{1/2}(z)\theta(x)\mathbf{1}_{\mathfrak{O}^*}(z)\tau\left(p\begin{pmatrix} zI_{n-1} & \\ & 1 \end{pmatrix}\right)v_0$, for z in F^* , p in P_n , k in $G_n(\mathfrak{O})$, and x in F^n , is a preimage of v_0 in $(\Phi^+\tau)^{P_{n+1}(\mathfrak{O})}$. \square

Now we are able to prove the following property of Φ^- , that we will be of great use later.

Proposition 1.4. *If τ belongs to $\text{Alg}(P_n)$, then Φ^- maps $\tau^{P_n(\mathfrak{O})}$ surjectively onto $\tau_{(1)}^{P_{n-1}(\mathfrak{O})}$, and Ψ^- maps $\tau^{G_{n-1}(\mathfrak{O})}$ surjectively onto $\tau_{(1)}^{G_{n-1}(\mathfrak{O})}$.*

Proof. For the first part, we use the filtrations $0 \subset \tau_{n-1} \subset \dots \subset \tau_0 = \tau$ of τ , and $0 \subset \tau_{(1),n-1} \subset \dots \subset \tau_{(1),0} = \tau_{(1)}$ of $\tau_{(1)}$. But τ_i equals $\Phi^+(\tau_{(1),i-1})$ because of Lemma 1.1, so that Φ^- maps $\tau_i^{P_n(\mathfrak{O})}$ onto $\tau_{(1),i-1}^{P_{n-1}(\mathfrak{O})}$ surjectively according to Proposition 1.3. In particular, Φ^- maps $\tau_1^{P_n(\mathfrak{O})}$ onto $\tau_{(1)}^{P_{n-1}(\mathfrak{O})}$ surjectively.

Ψ^- maps $\tau^{G_{n-1}(\mathfrak{O})}$ surjectively onto $\tau_{(1)}^{G_{n-1}(\mathfrak{O})}$ because $G_{n-1}(\mathfrak{O})$ is compact. \square

2 Mirabolic restriction for Whittaker functions

We start by recalling Proposition 2.1 of [C-P], which gives an interpretation of Φ^- in terms of restriction of Whittaker functions.

Proposition 2.1. *For any submodule τ of $(C^\infty(N_k \backslash P_k, \theta), \rho)$ (where ρ denotes the action of P_k by right translation), the map $R : W \mapsto \delta_{U_k}^{-1/2}W|_{P_{k-1}}$ is P_{k-1} -equivariant from $(C^\infty(N_k \backslash P_k, \theta), \rho)$ to $(C^\infty(N_{k-1} \backslash P_{k-1}, \theta), \rho)$, with kernel $\tau(\dot{U}_k, \theta)$. Hence it induces a P_{k-1} -modules isomorphism between $\Phi^-\tau$ and $\text{Im}(R) \subset C^\infty(N_{k-1} \backslash P_{k-1}, \theta)$, so that $(\text{Im}(R), \rho)$ is a model for $\Phi^-\tau$.*

Notice that for $k \geq 2$, if $g \in G_{k-1}$ equals pzk with $p \in P_{k-1}$, $z \in Z_{k-1}$, and $k \in G_{n-1}(\mathfrak{O})$, then the absolute value of z depends only on g , so we can write it $|z(g)|_F$.

We now state a proposition that follows from the proofs of Propositions 2.3. and 2.7. of [C-P], about the interpretation of Ψ^- in terms of Whittaker functions.

Proposition 2.2. *Let τ be a P_k -submodule of $C^\infty(N_k \backslash P_k, \theta)$, and suppose that $\tau^{(1)}$ is a G_{k-1} -module with central character c . Then, for any W in τ , for any g in G_{k-1} , the quantity $c^{-1}(z)|z|^{-(1-k)/2}W(zg)$ is constant whenever z is in a punctured neighbourhood of zero (maybe depending on g) in Z_{k-1} , and the linear map $L : W \mapsto \lim_{z \rightarrow 0} c^{-1}(z)|z|^{(1-k)/2}W(z)$ has a kernel containing $\tau(U_k, 1)$. It thus induces a (nonzero by definition) Whittaker functional on $\tau^{(1)}$.*

In the previous proposition, if we add the assumption that $\tau^{(1)}$ is irreducible (which guarantees the existence of the central character c), then it is known that a Whittaker functional L is unique up to scaling by an element of \mathbb{C}^* , moreover the map $v \in \tau^{(1)} \mapsto [g \mapsto L(\tau^{(1)}(g)v)] \in C^\infty(N_{k-1} \backslash G_{k-1}, \theta)$ is injective, and one can talk of the Whittaker model of $\tau^{(1)}$ with respect to θ (which is the image of the preceding map, and no reference to L is needed). We have the following corollary.

Corollary 2.1. *Let τ be a P_k -submodule of $(C^\infty(N_k \backslash P_k, \theta), \rho)$, and suppose that $\tau^{(1)}$ is an irreducible G_{k-1} -module with central character c , then it is generic and Ψ^- identifies with the map*

$$F : W \mapsto [g \mapsto \lim_{z \rightarrow 0} c^{-1}(z) |z|^{(1-k)/2} W(zg) \delta_{U_k}^{-1/2}(g)]$$

from τ to the Whittaker model $W(\tau^{(1)}, \theta)$ of $\tau^{(1)}$.

Proof. For W in τ , call \overline{W} its image in $\tau^{(1)}$, and call F_W the function

$$[g \mapsto \lim_{z \rightarrow 0} c^{-1}(z) |z|^{(1-k)/2} W(zg) \delta_{U_k}^{-1/2}(g)]$$

in $C^\infty(N_{k-1} \backslash G_{k-1}, \theta)$, then from Proposition 2.2 and the discussion after, the \mathbb{C} -linear map $F : W \mapsto F_W$ induces a \mathbb{C} -linear isomorphism $\overline{F} : \overline{W} \mapsto F_W$ between $\tau^{(1)}$ and its image in $(C^\infty(N_{k-1} \backslash G_{k-1}, \theta), \rho)$. Moreover, it is a G_{k-1} -equivariant because for $x \in G_{k-1}$, one has

$$\overline{F}(\tau^{(1)}(x)\overline{W}) = \overline{F}(\delta_{U_k}^{-1/2}(x)\overline{\rho(x)W}) = \delta_{U_k}^{-1/2}(x)F(\rho(x)W) = \rho(x)F(W) = \rho(x)\overline{F}(\overline{W}).$$

□

We end this section by stating two technical lemmas about Whittaker functions fixed under a maximal compact subgroup, the first is inspired from Lemma 9.2 of [J-P-S 2].

Lemma 2.1. *Let τ be a P_n -submodule of $C^\infty(N_n \backslash P_n, \theta)$, for $n \geq 3$, and let W belong to $\tau^{P_n(\mathfrak{O})}$, then there exists W' in $\tau^{P_n(\mathfrak{O})}$, such that $W'(p\beta_{n-1}(z)) = W(p)\mathbf{1}_{\mathfrak{O}^*}(z)$ for p in P_{n-1} and z in F^* .*

Proof. For l in \mathbb{Z} , we denote by ϕ_l the characteristic function $\mathbf{1}_{\mathfrak{P}^l}$ of \mathfrak{P}^l . In particular its Fourier transform $\widehat{\phi}_l^\theta$ with respect to θ is equal to $\lambda_l \phi_{-l}$ for some positive λ_l . We denote by Φ_l the function $\otimes_{i=1}^{n-1} \phi_l$, which is the characteristic of the lattice $\varpi^l \mathfrak{O}^{n-1}$ in F^{n-1} . We denote by u the natural isomorphism between F^{n-1} and U_n . We also recall that any element of τ is determined by its restriction to G_{n-1} .

We set $W^l(p) = \int_{x \in F^{n-1}} W(pu(x)) \Phi_l(x) dx$ for p in P_n , hence W^l belongs to τ . Moreover if k belongs to $G_{n-1}(\mathfrak{O})$, and g belongs to G_{n-1} , then $W^l(gk)$ is equal to $\int_{x \in F^{n-1}} W(gku(x)) \Phi_l(x) dx = \int_{x \in F^{n-1}} W(gu(kx)) \Phi_l(x) dx$ because W is $P_n(\mathfrak{O})$ -invariant, and this last integral is equal to $\int_{x \in F^{n-1}} W(gu(x)) \Phi_l(k^{-1}x) dx = \int_{x \in F^{n-1}} W(gu(x)) \Phi_l(x) dx = W^l(g)$ because of the invariance of dx and Φ_l under $G_{n-1}(\mathfrak{O})$. It is also clear that W^l is invariant $U_n(\mathfrak{O})$ because W is, hence W^l belongs to $\tau^{P_n(\mathfrak{O})}$.

Now one checks that, for p in P_{n-1} , and z in F^* , we have $W^l(p\beta_{n-1}(z)) = \lambda_l^{n-2} W(p) \widehat{\phi}_l^\theta(z) = \lambda_l^{n-1} W(p) \phi_{-l}(z)$. It is then clear that $W' = W^0/\lambda_0^{n-1} - W^{-1}/\lambda_{-1}^{n-1}$ satisfies the wanted properties.

□

Lemma 2.2. *Let τ be a P_n -submodule of $C^\infty(N_n \backslash P_n, \theta)$, for $n \geq 3$, and let W belong to $\tau^{G_{n-1}(\mathfrak{O})}$, then there exists W' in $\tau^{P_n(\mathfrak{O})}$, such that $W'(z_1, \dots, z_{n-1}, 1) = W(z_1, \dots, z_{n-1}, 1) \mathbf{1}_{\mathfrak{O}}(z_{n-1})$ for z_i in F^* .*

Proof. Set $W'(g) = \int_{u \in U_n(\mathfrak{O})} W(gu) du$.

□

3 Construction of the essential Whittaker function

We are now able to produce the essential vector of a generic representation π . Let's write π as a product of non-linked segments (see [Z], section 9).

$$\pi = [\nu^{-k_1(\pi)-1} \rho_1(\pi), \dots, \rho_1(\pi)] \times \dots \times [\nu^{-k_t(\pi)-1} \rho_t(\pi), \dots, \rho_t(\pi)],$$

where $\rho_i(\pi)$ is a cuspidal representation of $G_{r_i(\pi)}$ and one has $\sum_i k_i(\pi) r_i(\pi) = n$.

We recall that we denote by χ_1, \dots, χ_r the elements of the set $\{j, \rho_j(\pi) \in \widehat{F^*}/\mathfrak{D}^*\}$, counted with multiplicity. We denote by π_u the representation $\chi_1 \times \dots \times \chi_r$ of G_r . We recall that we only need to focus on the case $r \leq n-1$.

We first make the following assumption on π .

Assumption 1. *The representation π_u occurs as an irreducible submodule of $\pi^{(n-r)}$.*

In this case, the representation π_u is generic and admits a unique normalised spherical Whittaker function $W_{\pi_u}^0$ in $W(\pi_u, \theta)$.

We also notice the following facts. First, from the theory of Kirillov models (see [B-Z], theorem 4.9), the map $W \in W(\pi, \theta) \mapsto W|_{P_n}$ is injective, we denote by $W(\pi_{(0)}, \theta)$ its image. We choose this notation because P_n -module $\pi_{(0)} = \pi|_{P_n}$ is isomorphic to the submodule $W(\pi_{(0)}, \theta)$ of $(C^\infty(N_n \backslash P_n, \theta), \rho)$. Now if one applies proposition 2.1 repeatedly to $\pi_{(0)}$, then for $r \leq n-2$, the P_{r+1} -module $\pi_{(n-r-1)}$ is isomorphic to the submodule of $(C^\infty(N_{r+1} \backslash P_{r+1}, \theta), \rho)$, whose vectors are the functions $(\prod_{k=r+2}^n \delta_{U_k}^{-1/2}) W|_{P_{r+1}}$, for $W \in W(\pi, \theta)$, we denote by $W(\pi_{(n-r-1)}, \theta)$ this P_{r+1} -module.

The following proposition then holds.

Proposition 3.1. *There exists in $W(\pi_{(n-r-1)}, \theta)^{P_{r+1}(\mathfrak{D})}$ an element W , such that $W(z_1, \dots, z_r, 1) = \delta_{U_{r+1}}^{1/2}(z_1, \dots, z_r) W_{\pi_u}^0(z_1, \dots, z_r) 1_{\mathfrak{D}}(z_r)$ for z_r in F^* .*

Proof. By the second part of Proposition 1.4, there is an vector W' in $W(\pi_{(n-r-1)}, \theta)^{G_r(\mathfrak{D})}$ such that $\psi^- W = W_{\pi_u}^0$. But then, by the claim in the proof of Theorem 2.1 of [M], There is N in \mathbb{Z} , such that $W'(z_1, \dots, z_r a, 1) = c_{\pi_u}(a) |a|^{r/2} W'(z_1, \dots, z_r, 1)$ (parametrizing A_{r+1} with the β_i 's) for $|z_r| \leq q^{-N}$ and $|a| \leq 1$. For b in F^* , call W'_b the function $p \mapsto W'(p\beta_r(b))/(c_{\pi_u}(b)|b|^{r/2})$, then W'_b still belongs to $W(\pi_{(n-r-1)}, \theta)^{G_r(\mathfrak{D})}$, and $W'_b(z_1, \dots, z_r, 1)/(c_{\pi_u}(z_r)|z_r|^{r/2})$ is constant with respect to z_r whenever $|z_r| \leq q^{-N}/|b|$. Choosing b in F^* satisfying $|b| = q^{-N}$, then the function $W'_b(z_1, \dots, z_r, 1)/(c_{\pi_u}(z_r)|z_r|^{r/2})$ is constant with respect to z_r whenever $|z_r| \leq 1$. But then applying Lemma 2.2, there is W in $W(\pi_{(n-r-1)}, \theta)^{P_{r+1}(\mathfrak{D})}$, such that $W(z_1, \dots, z_r, 1)$ is equal to $W'_b(z_1, \dots, z_r, 1) 1_{\mathfrak{D}}(z_r)$. Moreover according to Corollary 2.1, the function W'_b , hence W satisfies $W(z_1, \dots, z_r, 1) = W'_b(z_1, \dots, z_r, 1) = |z_r|^{r/2} \delta_{U_{r+1}}^{1/2}(z_1, \dots, z_{r-1}) W_{\pi_u}^0(z_1, \dots, z_r)$ for $|z_r| \leq 1$. As $\delta_{U_{r+1}}(z_r) = |z_r|^r$, it is now clear that W has the desired property. \square

We now prove the existence of the essential Whittaker function under our assumption on π :

Theorem 3.1. *Let π be a generic representation of G_n satisfying Assumption 1. Then there exists in $W(\pi, \theta)$ a unique $G_{n-1}(\mathfrak{D})$ -invariant function W_{π}^{ess} , such that for every unramified generic representation π' of G_{n-1} , with normalised spherical function $W_{\pi'}^0$ in $W(\pi', \theta^{-1})$, one has the equality $I(W_{\pi}^{ess}, W_{\pi'}^0, s) = L(\pi, \pi', s)$.*

Proof. Uniqueness is known. For the existence, we suppose $r \leq n-1$.

We already constructed in the previous proposition an vector W in $W(\pi_{(n-r-1)}, \theta)^{P_{r+1}(\mathfrak{D})}$ such that $W(z_1, \dots, z_r, 1) = \delta_{U_{r+1}}^{1/2}(z_1, \dots, z_r) W_{\pi_u}^0(z_1, \dots, z_r) 1_{\mathfrak{D}}(z_r)$. Then, applying proposition 1.4 and then Lemma 2.1, we obtain W_1 in $W(\pi_{(n-r)}, \theta)^{P_{r+2}(\mathfrak{D})}$, that satisfies

$$\begin{aligned} W_1(z_1, \dots, z_{r+1}, 1) &= \delta_{U_{r+2}}^{1/2}(z_1, \dots, z_{r+1}) W(z_1, \dots, z_r, 1) 1_{\mathfrak{D}^*}(z_{r+1}) \\ &= \delta_{U_{r+2}}^{1/2}(z_1, \dots, z_r, 1) W(z_1, \dots, z_r, 1) 1_{\mathfrak{D}^*}(z_{r+1}) \\ &= \delta_{U_{r+2}}^{1/2}(z_1, \dots, z_r, 1) \delta_{U_{r+1}}^{1/2}(z_1, \dots, z_r) W_{\pi_u}^0(z_1, \dots, z_r) 1_{\mathfrak{D}}(z_r) 1_{\mathfrak{D}^*}(z_{r+1}). \end{aligned}$$

Repeating this last step, we obtain W_{π}^{ess} in $W(\pi, \theta)_n^P(\mathfrak{D})$, satisfying

$$W_{\pi}^{ess}(z_1, \dots, z_{n-1}, 1) = W_{\pi_u}^0(z_1, \dots, z_r) 1_{\mathfrak{D}}(z_r) \prod_{i=r+1}^n 1_{\mathfrak{D}^*}(z_i) \delta_{U_i}^{1/2}(z_1, \dots, z_r, \underbrace{1, \dots, 1}_{i-(r+1) \times}).$$

Hence, to conclude, according to the introduction, we only need to check that the product

$$\prod_{i=r+1}^n \delta_{U_i}^{1/2}(z_1, \dots, z_r, 1, \dots, 1)$$

is equal to

$$\frac{\delta_{N_{n-1}}(z_1, \dots, z_r, 1, \dots, 1)}{\delta_{N_r}(z_1, \dots, z_r)} |det(z_1, \dots, z_r)|^{\frac{r-n+2}{2}}.$$

We first notice that we have $\delta_{U_i}(z_1, \dots, z_r, 1, \dots, 1) = |det(z_1, \dots, z_r)|$ for $i > r$, which gives

$$\prod_{i=r+1}^n \delta_{U_i}^{1/2}(z_1, \dots, z_r, 1, \dots, 1) = |det(z_1, \dots, z_r)|^{\frac{n-r}{2}}.$$

But equation (1) implies that the equality (writing $det(z)$ instead of $det(z_1, \dots, z_r)$)

$$\frac{\delta_{N_{n-1}}(z_1, \dots, z_r, 1, \dots, 1)}{\delta_{N_r}(z_1, \dots, z_r)} = \prod_{i=r+1}^{n-1} \delta_{U_i}(z_1, \dots, z_r, 1, \dots, 1) = |det(z)|^{n-1-r},$$

we thus obtain the wanted identity

$$\frac{\delta_{N_{n-1}}(z_1, \dots, z_r, 1, \dots, 1)}{\delta_{N_r}(z_1, \dots, z_r)} |det(z)|^{\frac{r-n+2}{2}} = |det(z)|^{n-1-r+\frac{r-n+2}{2}} = |det(z)|^{\frac{n-r}{2}}.$$

□

Let's try to get rid of our assumption.

We first notice the following fact. Suppose that the representations π and π^\vee satisfy the assumption, then it is proved in Theorem 5 of [J-P-S] (which is valid as soon as W_π^{ess} and $W_{\pi^\vee}^{ess}$ satisfy the property of computing Rankin-Selberg L -functions as in Theorem 3.1), that W_π^{ess} is fixed under the open subgroup $H_n(m(\pi))$ of G_n defined in the introduction, where $m(\pi)$ is the conductor of π (i.e. the integer m such that the Godement-Jacquet epsilon factor $\epsilon(\pi, X)$ is equal to cX^m).

If π is generic, as we said in the introduction, we write π as the product

$$\pi = [\nu^{-k_1(\pi)-1}\rho_1(\pi), \dots, \rho_1(\pi)] \times \dots \times [\nu^{-k_t(\pi)-1}\rho_t(\pi), \dots, \rho_t(\pi)],$$

and put

$$\pi^v = \nu^{v_1}[\nu^{-k_1(\pi)-1}\rho_1(\pi), \dots, \rho_1(\pi)] \times \dots \times \nu^{v_t}[\nu^{-k_t(\pi)-1}\rho_t(\pi), \dots, \rho_t(\pi)]$$

for $v = (v_1, \dots, v_t)$ in \mathcal{D}^t .

Then for every v such that π^v and π^{v^\vee} satisfy our assumption, it will be important to us, to notice that the essential function $W_{\pi^v}^{ess}$, which is well defined, is fixed under $H_n(m(\pi))$, because $m(\pi^v)$ doesn't depend on v . Indeed, from Theorem 3.4 of [G-J], one has $\epsilon(\pi, X) = \prod_{i=1}^t \epsilon(\nu^{v_i}[\nu^{-k_i(\pi)-1}\rho_1(\pi), \dots, \rho_i(\pi)], X)$, and each $\epsilon(\nu^{v_i}[\nu^{-k_i(\pi)-1}\rho_1(\pi), \dots, \rho_i(\pi)], X)$ is equal to $q^{-v_i} \epsilon([\nu^{-k_i(\pi)-1}\rho_1(\pi), \dots, \rho_i(\pi)], X)$, so that $\epsilon(\pi^v, X)$ is a multiple of $\epsilon(\pi, X)$.

We now recall that all π^v can be realised in a same vector space V_π^{comp} , the so called compact realisation of π , which is given by restriction to $G_n(\mathfrak{O})$ of functions in π , when one sees π as a space of functions from G_n to $\otimes_i W([\nu^{-k_i(\pi)-1}\rho_i(\pi), \dots, \rho_i(\pi)], \theta)$. One has a vector space isomorphism $f \mapsto f^v$ from V_π^{comp} to π^v , such that for fixed g in G_n , and f in V_π^{comp} , the map $v \mapsto f^v(g)$ is polynomial in $v \in \mathcal{D}^t$.

It is well known (for instance [C-P], Section 4), that for every f in V_π^{comp} , one can build a Whittaker function $W_{f,v} = W_{f^v}$ in $W(\pi^v, \theta)$ (the image of the up to scaling only nonzero intertwining operator from π^v to $\mathcal{C}^\infty(N_n \backslash G_n, \theta)$), and that again, $f \mapsto W_{f,v}$ is a surjective linear map from V_π^{comp} to $W(\pi^v, \theta)$, which is an isomorphism whenever π^v is generic (i.e. irreducible). In this case, one shows again that for fixed g in G_n , and f in V_π^{comp} , the map $v \mapsto W_{f,v}(g)$ is polynomial on \mathcal{D}^t .

By an (affine) hyperplane of \mathcal{D}^t , we mean a set $H_{a,c} = \{v \in \mathcal{D}^t, q^{<a,v>} = c\}$ for $a = (a_i)_i \in \mathbb{Z}^t - \{0\}$, $c \in \mathbb{C}^*$, and $<a,v> = \sum_{i=1}^t a_i v_i$. The first step is to prove that Assumption 1 is verified for v outside a finite number of such hyperplanes:

Proposition 3.2. *For v in outside finite number of affine hyperplanes $(H_{a_k, c_k})_{k=1, \dots, l}$ of \mathcal{D}^t , Assumption 1 holds for π^v and π^{v^\vee} , and both are irreducible (hence generic).*

Proof. Indeed, according to Proposition 4.5 of [C-P], for v outside a finite number of hyperplanes \mathcal{D}^t , π^v as well as all its derivatives are semi-simple, and the same holds for π^{v^\vee} . Hence for v outside a finite number hyperplanes, both π^v and π^{v^\vee} are irreducible, with completely reducible derivatives, in particular Assumption 1 holds for both. \square

Now we choose a vector $v_0 \in \mathbb{Z}^t \subset \mathbb{C}^t$, whose image in $\mathcal{D}^t = \mathbb{C}^t / (\frac{2i\pi}{Ln(q)}\mathbb{Z})^t$ is not parallel to any of the H_k 's (i.e. $q^{<a_k, v_0>} \neq 1$ for k in $\{1, \dots, l\}$), then the line $\{uv_0 \in \mathcal{D}^t, u \in \mathcal{D}\}$, meets H_k at most $|<a_k, v_0>|$ times, hence it meets $\bigcup_{k=1}^l H_k$ for u in a finite set $\{u_1, \dots, u_m\}$. We denote by $v_0(u)$ the element uv_0 of \mathcal{D}^t .

We now prove that if π is generic, the map $u \mapsto W_{\pi^{v_0(u)}}^{ess}$ which is defined for all u outside $\{u_1, \dots, u_m\}$, has a limit when u tends to zero, which is the essential vector of π .

Theorem 3.2. *Let π be a generic representation of G_n . For g in G_n , the map $u \mapsto W_{\pi^{v_0(u)}}^{ess}(g)$ defined on the Zariski open subset $U = \mathcal{D} - \{u_1, \dots, u_m\}$ of \mathcal{D} is regular, and has a limit $W_{\pi^0}^{ess}(g)$ at $u = 0$, which defines a $P_n(\mathfrak{O})$ -invariant function in $W(\pi, \theta)$.*

Moreover, the function $W_{\pi^0}^{ess}$ satisfies $I(W_{\pi^0}^{ess}, W_{\pi^0}^0, s) = L(\pi^0, \pi', s)$ for any generic unramified representation π' of G_{n-1} . In particular, $W_{\pi^0}^{ess}$ is the essential vector of $W(\pi, \theta)$.

Proof. Let's denote by $\mathbb{C}[\mathcal{D}]$ the \mathbb{C} -algebra of polynomial functions on the complex algebraic variety \mathcal{D} , and by $\mathbb{C}[U]$ the image of $\mathbb{C}[\mathcal{D}]$ under the restriction map $r_U : f \mapsto f|_U$, so that r_U extends to a \mathbb{C} -algebra isomorphism between the fraction fields $\mathbb{C}(\mathcal{D})$ and $\mathbb{C}(U)$, we will thus identify elements of $\mathbb{C}(\mathcal{D})$ and $\mathbb{C}(U)$.

For u in U , we know that $W_{\pi^{v_0(u)}}^{ess}$ belongs to $W(\pi^{v_0(u)}, \theta)^{H_n(m)}$, for $m = m(\pi)$. Hence if f_1, \dots, f_d is a basis of $(V_\pi^{comp})^{H_n(m)}$, the functions $W_{f_i, v_0(u)}$ form a basis of $W(\pi^{v_0(u)}, \theta)^{H_n(m)}$. So for u in U , there are d complex numbers $\lambda_{i,u}$ such that $W_{\pi^{v_0(u)}}^{ess} = \sum_{i=1}^d \lambda_{i,u} W_{f_i, v_0(u)}$.

But for p in P_n , and because of the Iwasawa decomposition ($p = nak$, with $k \in G_{n-1}(\mathfrak{O})$) and the explicit description of the restriction of $W_{\pi^{v_0(u)}}^{ess}$ to the torus A_n for u in U , the map $u \mapsto W_{\pi^{v_0(u)}}^{ess}(p)$ belongs to $\mathbb{C}[U]$. The maps $u \mapsto W_{f_i, v_0(u)}(p)$ also belong to $\mathbb{C}[U]$ for p in P_n (in fact for any p in G_n).

We thus can consider the functions $W_{f_i} : p \mapsto [u \mapsto W_{f_i, v_0(u)}(p)]$ from P_n to $\mathbb{C}[U] \subset \mathbb{C}(U)$, they form the basis of a d -dimensional $\mathbb{C}(U)$ -vector subspace M of $\mathcal{C}^\infty(N_n \backslash P_n, \theta, \mathbb{C}(U)) = Ind_{N_n}^{P_n}(\theta, \mathbb{C}(U))$ (indeed, a dependance relation $\sum a_i W_{f_i} = 0$ over $\mathbb{C}(U)$, gives for all u in U , except a finite number, the relation $\sum a_i(u) (W_{f_i, v_0(u)})|_{P_n} = 0$ in $W(\pi^{v_0(u)}, \theta)$, which implies that $a_i(u)$ is zero, as restriction to P_n is injective on $W(\pi^{v_0(u)}, \theta)$). Now the restrictions to M of the linear forms in $Hom_{\mathbb{C}(U)}(\mathcal{C}^\infty(N_n \backslash P_n, \theta, \mathbb{C}(U)), \mathbb{C}(U))$ of the type $Ev_p : W \mapsto W(p)$ span the $\mathbb{C}(U)$ -vector space $Hom_{\mathbb{C}(U)}(M, \mathbb{C}(U))$, because the intersection of their kernels is zero. In particular, if (L_1, \dots, L_d) is the dual basis in $Hom_{\mathbb{C}(U)}(M, \mathbb{C}(U))$ of the basis $(W_{f_1}, \dots, W_{f_d})$ of M (i.e. $L_i(W_{f_j})$ is the Kronecker symbol δ_i^j), we can write L_i as $\sum_{k=1}^d a_{k,i} Ev_{p_k}|_M$ for elements p_1, \dots, p_d of P_n , and $a_{k,i}$'s in $\mathbb{C}(U)$.

We denote by W_{ess} the vector $p \mapsto [u \mapsto W_{\pi^{v_0(u)}}^{ess}(p)]$ of $\mathcal{C}^\infty(N_n \backslash P_n, \theta, \mathbb{C}(U))$, by λ_i the element

$\sum_{k=1}^d a_{k,i} E_{p_k}(W_{ess})$ of $\mathbb{C}(U)$, and by $P(U)$ the finite subset of U consisting of the poles of the functions $a_{k,i}$ for $(k,i) \in \{1, \dots, d\}^2$.

For u in $U - P(U)$, we have the identities:

$$\begin{aligned} \lambda_{i_0}(u) &= \sum_{k=1}^d a_{k,i_0}(u) W_{\pi^{v_0}(u)}^{ess}(p_k) = \sum_{k=1}^d a_{k,i_0}(u) (\sum_{i=1}^d \lambda_{i,u} W_{f_i,u}(p_k)) \\ &= \sum_{i=1}^d \lambda_{i,u} (\sum_{k=1}^d a_{k,i_0}(u) W_{f_i,u}(p_k)) = \sum_{i=1}^d \lambda_{i,u} L_{i_0}(W_{f_i})(u) \\ &= \lambda_{i_0,u} \end{aligned}$$

Now set for each g in G_n , $W_u(g) = \sum_{i=1}^d \lambda_i(u) W_{f_i,u}(g)$ in $\mathbb{C}(\mathcal{D})$. If u doesn't belong to the set $P'(\mathcal{D})$ of the poles of the λ_i 's in \mathcal{D} , this defines an element W_u in $W(\pi^{v_0}(u), \theta)^{H_n(m)}$, and we just saw that $W_u = W_{\pi^{v_0}(u)}^{ess}$ for u in $U - P(U)$, which implies that for all p in P_n , one has $W_u(p) = W_{\pi^{v_0}(u)}^{ess}(p)$ as elements of $\mathbb{C}(\mathcal{D})$ (in particular $W_u(p)$ belongs to $\mathbb{C}[\mathcal{D}]$).

We're going to see $P'(\mathcal{D})$ doesn't intersect the set of elements u in \mathcal{D} , such that $\pi^{v_0}(u)$ is generic (i.e. irreducible), in particular, W_0 will define an element of $W(\pi, \theta)^{H_n(m)}$, and the λ_i 's (hence $W_u(g)$ for g in G_n) are in fact in $\mathbb{C}[U]$.

We just deal with $u = 0$, the same applies for the other elements u in \mathcal{D} such that $\pi^{v_0}(u)$ is generic. If one of the λ_i 's had a pole at zero, then there would be $m \geq 1$, such that $(q^u - 1)^m \lambda_i(u)$ would have no pole at 0 for every i (so that $(q^u - 1)^m W_u = \sum_{i=1}^d (q^u - 1)^m \lambda_i(u) W_{f_i,u}$ defines an element W in $W(\pi, \theta)^{H_n(m)}$ when $u = 0$), and there would be i_1 between 1 and d , such that $(q^u - 1)^m \lambda_{i_1}(u)$ tends to a nonzero complex number c when u tends to zero. But then, as $W_u(p)$ belongs to $\mathbb{C}[\mathcal{D}]$ for p in P_n , and setting $\mu_i(u) = (q^u - 1)^m \lambda_i(u)$, one would deduce $\sum_{i=1}^d \mu_i(0) W_{f_i,0}(p) = 0$, and $\sum_{i=1}^d \mu_i(0) W_{f_i,0}$ would be zero in $W(\pi, \theta)$, which is not possible as $\mu_{i_1}(0) \neq 0$, and the vectors $W_{f_i,0}$ are independant over \mathbb{C} .

Let's recall the properties of W_u we have obtained so far: for all g in G_n , $W_u(g)$ is an element of $\mathbb{C}(\mathcal{D})$ which is regular for u in $V = U \cup \{0\}$, and it defines an element W_u in $W(\pi^{v_0}(u), \theta)^{H_n(m)}$ for u in V . Moreover, for p in P_n , $W_u(p) = W_{\pi^{v_0}(u)}^{ess}(p) \in \mathbb{C}[\mathcal{D}]$ (hence for all u in U , $W_u = W_{\pi^{v_0}(u)}^{ess}$). Conclusion: for u in V , it follows from Proposition 4.2 of [C-P] that if π' a generic unramified representation of G_{n-1} , the integral $I(W_u, W_{\pi'}^0, s) = \sum_{i=1}^d \lambda_i(u) I(W_{f_i,u}, W_{\pi'}^0, s)$ is absolutely convergent when $\text{Re}(\Lambda(u, s)) > 0$ for some linear form Λ with real coefficients, and defines an element of $\mathbb{C}(\mathcal{D}^2)$. But as the map $L(\pi^{v_0}(u), \pi', s)$ is also rational on \mathcal{D}^2 , and as $I(W_u, W_{\pi'}^0, s)$ and $L(\pi^{v_0}(u), \pi', s)$ match on the Zariski open $U \times \mathcal{D}$ of \mathcal{D}^2 , they are equal, and we obtain the desired equality for $u = 0$. □

Remark 3.1. It is clear from our description of the essential function of a generic representation π of G_n , that for any spherical generic π' representation of G_m , for $m < n$, the Rankin-Selberg integral $I(W_{\pi}^{ess}, W_{\pi'}^0, s)$ is equal to $L(\pi, \pi', s)$.

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